

# MODULUS OF SUPPORTING CONVEXITY AND SUPPORTING SMOOTHNESS

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**ABSTRACT.** We introduce the moduli of the supporting convexity and the supporting smoothness of a Banach space, which characterize the deviation of the unit sphere from an arbitrary supporting hyperplane. We show that the modulus of supporting smoothness, the Banaś modulus, and the modulus of smoothness are all equivalent at zero, the modulus of supporting convexity is equivalent at zero to the modulus of convexity. We prove a Day–Nordlander type result for these moduli.

## 1. INTRODUCTION

The properties of a Banach space are completely determined by its unit ball. The geometry of the unit ball of a Banach space  $X$  may be described, for instance, using the properties of some moduli attached to  $X$ . (For example, the moduli of convexity, of smoothness, Milman's moduli, etc.) The aim of this paper is to introduce and explore some new type of moduli, which characterize the deviation of the unit sphere from an arbitrary supporting hyperplane.

In the sequel we shall need some additional notation. Let  $X$  be a real Banach space. For a set  $A \subset X$  by  $\partial A$ ,  $\text{int } A$  we denote the boundary and the interior of  $A$ . We use  $\langle p, x \rangle$  to denote the value of a functional  $p \in X^*$  at a vector  $x \in X$ . For  $R > 0$  and  $c \in X$  we denote by  $\mathfrak{B}_R(c)$  the closed ball with center  $c$  and radius  $R$ , by  $\mathfrak{B}_R^*(c)$  we denote the ball in the conjugate space. By definition, put  $J_1(x) = \{p \in \partial \mathfrak{B}_1^*(o) : \langle p, x \rangle = \|x\|\}$ . For convenience, the length of segment  $ab$  is denoted by  $\|ab\|$ , i.e.,  $\|ab\| = \|a - b\|$ .

We say that  $y$  is *quasiorthogonal* to the vector  $x \in X \setminus \{o\}$  and write  $y \perp x$  if there exists a functional  $p \in J_1(x)$  such that  $\langle p, y \rangle = 0$ . Note that the following conditions are equivalent:

- $y$  is quasiorthogonal to  $x$
- for any  $\lambda \in \mathbb{R}$  the vector  $x + \lambda y$  lies in the supporting hyperplane to the ball  $\mathfrak{B}_{\|x\|}(o)$  at  $x$ ;
- for any  $\lambda \in \mathbb{R}$  the following inequality holds  $\|x + \lambda y\| \geq \|x\|$ ;
- $x$  is orthogonal to  $y$  in the sense of Birkhoff–James ([6], Ch. 2, §1).

Let

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in \mathfrak{B}_1(o), \|x - y\| \geq \varepsilon \right\}$$

and

$$\rho_X(\tau) = \sup \left\{ \frac{\|x + y\|}{2} + \frac{\|x - y\|}{2} - 1 : \|x\| = 1, \|y\| = \tau \right\}.$$

The functions  $\delta_X(\cdot) : [0, 2] \rightarrow [0, 1]$  and  $\rho_X(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  are referred to as the moduli of convexity and smoothness of  $X$  respectively.

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Let  $f$  and  $g$  be two non-negative functions, each one defined on a segment  $[0, \varepsilon]$ . We shall consider  $f$  and  $g$  as *equivalent at zero*, denoted by  $f(t) \asymp g(t)$  as  $t \rightarrow 0$ , if there exist positive constants  $a, b, c, d, e$  such that  $af(bt) \leq g(t) \leq cf(dt)$  for  $t \in [0, e]$ .

The rest of this paper is organized as follows. In Section 2 we prove several simple technical lemmas, in Section 3 we introduce the definitions of the modulus of supporting convexity and the modulus of supporting smoothness and consider their basic properties, in Section 4 we show these modulus are equivalent at zero to the modulus of convexity and smoothness respectively, in Section 5 we prove that the moduli of smoothness, of supporting smoothness and the modulus of Banaś are all equivalent at zero, and, finally, in Section 6 we prove some estimates for these moduli concerning the maximal value of the Lipschitz constant for the metric projection operator onto a hyperplane.

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## 2. TECHNICAL RESULTS

In this section we prove several simple technical results.

The proof of the next lemma is trivial.

**Lemma 1.** *Suppose the set  $\mathfrak{B}_1(o) \setminus \text{int } \mathfrak{B}_r(o_1)$  is nonempty. Then it is arcwise connected.*

**Lemma 2.** *Let  $X_2$  be a two-dimensional Banach space. Suppose  $a, b, c, d \in \partial \mathfrak{B}_1(o)$  and the segments  $ab, cd$  intersect in point  $x$ . Then the following inequality holds*

$$\min\{\|cx\|, \|xd\|\} \leq \max\{\|ax\|, \|xb\|\}.$$

**Proof.**

Assume the converse. Then for some  $\varepsilon > 0$  we get  $\min\{\|cx\|, \|xd\|\} > \max\{\|ax\|, \|xb\|\} + \varepsilon = r$ . Since the segment  $ab$  belongs to  $\text{int } \mathfrak{B}_r(x)$  and separates it into two parts, then we cannot connect points  $c, d$  in  $\mathfrak{B}_1(o) \setminus \text{int } \mathfrak{B}_r(x)$ . This contradicts Lemma 1. The lemma is proved.  $\square$

**Lemma 3.** *Let  $x, y \in X$ ,  $x \neq o$ ,  $p \in \partial \mathfrak{B}_1^*(o)$  such that  $\langle p, x \rangle = \|x\|$ . Then*

$$(1) \quad \|x + y\| \leq \|x\| + \langle p, y \rangle + 2\|x\| \rho_X\left(\frac{\|y\|}{\|x\|}\right).$$

**Proof.**

By definition of the modulus of smoothness, we get

$$\frac{1}{2} \left( \frac{\|x + y\|}{\|x\|} + \frac{\|x - y\|}{\|x\|} \right) - 1 \leq \rho_X\left(\frac{\|y\|}{\|x\|}\right).$$

Multiplying both sides by  $2\|x\|$ , after some transformations we obtain:

$$\begin{aligned} \|x + y\| &\leq 2\|x\| - \|x - y\| + 2\|x\| \rho_X\left(\frac{\|y\|}{\|x\|}\right) \leq \\ 2\|x\| + \langle p, y - x \rangle + 2\|x\| \rho_X\left(\frac{\|y\|}{\|x\|}\right) &= \|x\| + \langle p, y \rangle + 2\|x\| \rho_X\left(\frac{\|y\|}{\|x\|}\right). \end{aligned}$$

$\square$

**Lemma 4.** For any vectors  $x, y \in X \setminus \{o\}$  the following inequality is true

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\|}.$$

**Proof.**

Using the triangle inequality, we get

$$\begin{aligned} \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| &= \left\| \left( \frac{x}{\|x\|} - \frac{y}{\|x\|} \right) + \left( \frac{y}{\|x\|} - \frac{y}{\|y\|} \right) \right\| \leq \left\| \left( \frac{x}{\|x\|} - \frac{y}{\|x\|} \right) \right\| + \left\| \left( \frac{y}{\|x\|} - \frac{y}{\|y\|} \right) \right\| \leq \\ &\leq \frac{1}{\|x\|} \|x - y\| + \|y\| \left\| \frac{1}{\|x\|} - \frac{1}{\|y\|} \right\| \leq \frac{2\|x - y\|}{\|x\|}. \end{aligned}$$

□

### 3. DEFINITIONS AND BASIC PROPERTIES

Let  $x, y \in \partial\mathfrak{B}_1(o)$  be such that  $y \lrcorner x$ . By definition, put

$$\lambda_X(x, y, r) = \min \{ \lambda \in \mathbb{R} : \|x + ry - \lambda x\| = 1 \}$$

for any  $r \in [0, 1]$ . Denote

$$\lambda_X^-(x, y, r) = \min \{ \lambda_X(x, y, r), \lambda_X(x, -y, r) \}; \quad \lambda_X^+(x, y, r) = \max \{ \lambda_X(x, y, r), \lambda_X(x, -y, r) \}.$$

**Definition 1.** For any  $r \in [0, 1]$  and  $x \in \partial\mathfrak{B}_1(o)$  we define the *modulus of local supporting convexity* as

$$\lambda_X^-(x, r) = \inf \lambda_X^-(x, y, t),$$

and respectively, the *modulus of local supporting smoothness* as

$$\lambda_X^+(x, r) = \sup \lambda_X^+(x, y, t),$$

where we choose  $(y, t)$  such that  $\|y\| = 1$ ,  $y \lrcorner x$ ,  $0 \leq t \leq r$  to minimize (maximize)  $\lambda_X^-(x, r)$  ( $\lambda_X^+(x, r)$ ).

It is clear that  $\lambda_X^-(x, r) \leq \lambda_X^+(x, r) \leq 1$ .

**Definition 2.** For any  $r \in [0, 1]$  we define the *modulus of supporting convexity* as

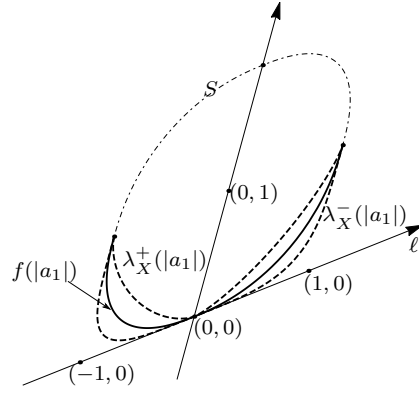
$$\lambda_X^-(r) = \inf \lambda_X^-(x, t),$$

and respectively, the *modulus of supporting smoothness* as

$$\lambda_X^+(r) = \sup \lambda_X^+(x, t),$$

where we choose  $(x, t)$  such that  $x \in \mathfrak{B}_1(o)$ ,  $0 \leq t \leq r$  to minimize (maximize)  $\lambda_X^-(r)$  ( $\lambda_X^+(r)$ ).

Let us explain the geometrical meaning of the moduli of supporting convexity and of supporting smoothness. Fix  $y, x \in \partial\mathfrak{B}_1(o)$  such that  $y \lrcorner x$ . Consider the plane  $L = \text{Lin}\{y, x\}$ . We use  $(a_1, a_2)$  to denote the vector  $a = a_1 y + a_2 x$  in this plane. The coordinate line  $\ell = \{(a_1, a_2) | a_1 \in \mathbb{R}, a_2 = 0\}$  is a tangent to the unit "circle"  $S = L \cap \partial\mathfrak{B}_1(o)$ . By the convexity of the ball, there is a convex function  $f : [-1, 1] \rightarrow \mathbb{R}$  such that for  $a_1 \in [-1, 1]$  the point  $(a_1, f(a_1))$  belongs to the lower semicircle of  $S$  (see Fig. 1). Hence for  $a_1 \in [-1, 1]$  the functions  $\lambda_X^-(|a_1|)$  and  $\lambda_X^+(|a_1|)$  are the lower and upper bounds to the  $f(a_1)$  respectively, i.e. the following inequalities hold  $\lambda_X^-(|a_1|) \leq f(a_1) \leq \lambda_X^+(|a_1|)$ .

FIGURE 1. Geometrical meaning of the  $\lambda_X^+(r), \lambda_X^-(r)$ .

**Lemma 5.** *Let  $X$  be an arbitrary Banach space, then:*

- (i)  $\lambda_X^+(0) = \lambda_X^-(0) = 0$ ;
- (ii) *for any  $r \in [0, 1]$  the following inequality holds:  $0 \leq \lambda_X^-(r) \leq \lambda_X^+(r) \leq r$ ;*
- (iii) *for any  $0 < r_1 < r_2 < 1$  we have*

$$(2) \quad \frac{r_2}{r_1} \lambda_X^-(r_1) \leq \lambda_X^-(r_2),$$

$$(3) \quad \lambda_X^-(r_2) - \lambda_X^-(r_1) \leq \frac{r_2 - r_1}{1 - r_1};$$

- (iv) *the modulus of supporting convexity is an increasing, continuous function on  $[0, 1]$  and moreover it is a strictly increasing function on the set  $\{r \in [0, 1] : \lambda_X^-(r) > 0\}$ ;*
- (v) *the modulus of supporting smoothness is a strictly increasing, convex and continuous function on  $[0, 1]$  and furthermore  $\lambda_X^+(1) = 1$ .*

**Proof.**

Let us introduce some notation. Fix  $x, y \in \partial \mathfrak{B}_1(o)$  such that  $y \lhd x$ , and real numbers  $r_1, r_2$  such that  $0 < r_1 < r_2 < 1$ . Let  $z = x + y$ ,  $z_i = x + r_i y$  where  $i = 1, 2$ . Let  $y_1, y_2 \in \partial \mathfrak{B}_1(o)$  such that  $y_i z_i \parallel ox$  and the intersection of the segment  $y_i z_i$  and the ball  $\mathfrak{B}_1(o)$  is the point  $y_i$  where  $i = 1, 2$ . (see Fig. 2). By construction  $\|y_i z_i\| = \lambda_X(x, y, r_i)$  where  $i = 1, 2$ . The reader will

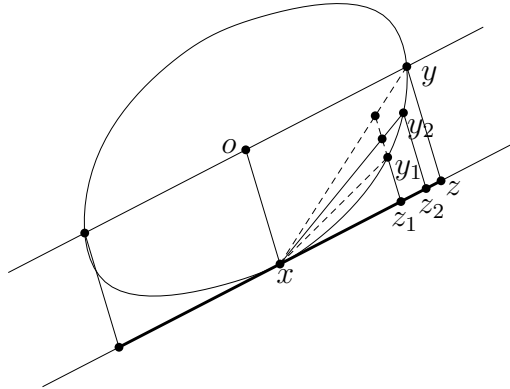


FIGURE 2. Illustration for Lemma 5.

have no difficulty in showing that it is enough to prove all the assertions of this Lemma for  $\lambda_X(x, y, r)$ . Now, let us prove the Lemma.

- (1) By the definitions, we have  $\lambda_X^+(0) = \lambda_X^-(0) = 0$ .
- (2) The first two inequalities of assertion (ii) are trivial. By similarity, we have  $\lambda_X(x, y, r) \leq r$ . Indeed,  $y_1 z_1 \parallel zy$  and  $y_1 z_1 \subset \triangle xyz$ . Taking the supremum we get assertion (ii).
- (3) Taking into account that  $\mathfrak{B}_1(o)$  is convex, we get  $y_1 z_1 \subset xy_2 z_2$ . By construction we have that  $y_1 z_1 \parallel z_2 y_2$ . By the similarity, we get  $\|y_2 z_2\| \geq \frac{r_2}{r_1} \|y_1 z_1\|$ , i.e.  $\frac{r_2}{r_1} \lambda_X(x, y, r_1) \leq \lambda_X(x, y, r_2)$ . Taking the infimum in  $\lambda_X(x, y, r_2)$ , we complete the proof of inequality (2).

By the convexity of the unit ball, we obtain that segment  $y_2 z_2$  lies in trapezoid  $y_1 z_1 zy$ . By construction  $y_2 z_2 \parallel y_1 z_1 \parallel yz$ . By similarity, we get

$$\|y_2 z_2\| - \|y_1 z_1\| \leq (1 - \|y_1 z_1\|) \frac{r_2 - r_1}{1 - r_1} \leq \frac{r_2 - r_1}{1 - r_1}.$$

Taking the infimum in  $\|y_1 z_1\| \rightarrow \lambda_X^-(r_1)$ , we have  $\|y_2 z_2\| - \lambda_X^-(r_1) \leq \frac{r_2 - r_1}{1 - r_1}$ . This yields (3).

- (4) Assertion (iv) is the direct consequence of assertion (iii).
- (5) The function  $\lambda_X^+(\cdot)$  is the supremum of the convex functions, therefore it's convex. Since  $\lambda_X^+(\cdot)$  is a convex bounded function and  $\lambda_X(x, y, r)$  is continuous in  $r$ , we obtain that  $\lambda_X^+(\cdot)$  is continuous on  $[0, 1]$ . We will prove that  $\lambda_X^+(r) > 0$  on  $(0, 1]$  in Lemma 7 below. By this and the equality  $\lambda_X^+(0) = 0$  and convexity of the modulus of supporting smoothness, we get that it is a strictly increasing function. The inequality  $\lambda_X^+(r) \leq r$  was proved in assertion (ii). The equality  $\lambda_X^+(1) = 1$  is the consequence of inequality (11) at  $r = 1$ , which will be proved below.

□

From Lemma 5 we have that in the definitions of the moduli of the supporting smoothness and supporting convexity one may choose  $t = r$ .

**Remark 1.** *Since any two plane central sections of the unit ball in a Hilbert space  $H$  are equal, we have*

$$\lambda_H^+(r) = \lambda_H^-(r) = \delta_H(2r) = 1 - \sqrt{1 - r^2}.$$

#### 4. COMPARISON OF SUPPORTING MODULI WITH THE MODULI OF CONVEXITY AND SMOOTHNESS

**Theorem 1.** *Let  $X$  be an arbitrary Banach space. Then  $\lambda_X^-(\varepsilon) \asymp \delta_X(\varepsilon)$  as  $\varepsilon \rightarrow 0$  and for any  $r \in [0; 1]$  :*

$$(4) \quad \delta_X(r) \leq \lambda_X^-(r) \leq \delta_X(2r).$$

**Proof.**

1) By the definition of the modulus of supporting convexity for any  $\varepsilon > 0$  there exists a parallelogram  $xyzd$  such that  $x, z \in \partial\mathfrak{B}_1(o)$ , the point  $d$  lies in the segment  $xo$  and  $\|xy\| = r$ ,  $xy \perp ox$ ,  $\|yz\| \leq \lambda_X^-(r) + \varepsilon$ . Therefore  $\|od\| = 1 - \|yz\|$ , consequently

$\delta_X(r) = \delta_X(\|zd\|) \leq \|yz\| \leq \lambda_X^-(r) + \varepsilon$ . Passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain the left-hand side of chain (4).

2) Let us prove the right-hand side of chain (4).

Fix  $r \in (0, 1)$  (if  $r = 0$  or  $r = 1$  the inequality is trivial). By the definition of the modulus of supporting convexity for any  $\varepsilon > 0$  there exist points  $a_\varepsilon, b_\varepsilon$  on the unit sphere such that  $\|a_\varepsilon b_\varepsilon\| \geq 2r$  and for the point  $c_\varepsilon = \frac{a_\varepsilon + b_\varepsilon}{2}$  the following inequality holds:

$$(5) \quad 1 - \|oc_\varepsilon\| \leq \delta_X(2r) + \varepsilon.$$

Let the ray  $oc_\varepsilon$  intersect the unit sphere in a point  $x$ . Denote by  $l_1$  the supporting line to the unit sphere such that  $l_1$  lies in the plane  $oa_\varepsilon b_\varepsilon$  and  $x \in l_1$ . Let  $l_2$  be a line such that  $l_1 \parallel l_2$  and  $c_\varepsilon \in l_2$ . Denote by  $f, g$  the points of intersections of  $\partial\mathfrak{B}_1(o)$  and the line  $l_2$ . From Lemma 2 it follows that  $\|f - c_\varepsilon\| \geq r$  or  $\|g - c_\varepsilon\| \geq r$ . Without loss of generality, put  $\|g - c_\varepsilon\| \geq r$ . Let  $l_3$  be a line such that  $l_3 \parallel oc_\varepsilon$  and  $g \in l_3$ . By definition, we put  $y = l_3 \cap l_1$  (see Fig. 3). Then

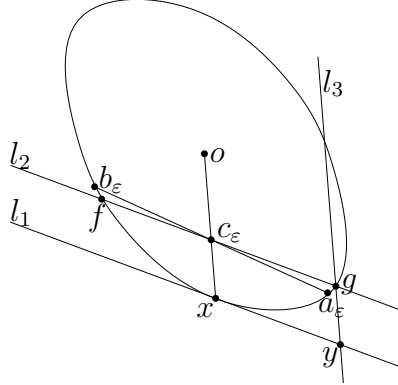


FIGURE 3. Illustration for Lemma 1.

$$\delta_X(2r) + \varepsilon \geq \|c_\varepsilon x\| \geq \lambda_X^-\left(x, \frac{y-x}{\|y-x\|}, \|y-x\|\right) \geq \lambda_X^-\left(x, \frac{y-x}{\|y-x\|}, r\right) \geq \lambda_X^-(r),$$

i.e.,  $\delta_X(2r) + \varepsilon \geq \lambda_X^-(r)$ . Passing to the limit as  $\varepsilon \rightarrow 0$ , we complete the proof.

□

**Lemma 6.** Let  $r \in [0, \frac{1}{2}]$ . Then

$$(6) \quad \lambda_X^+(r) \leq \rho_X(2r).$$

**Proof.**

Denote  $\lambda = \lambda_X^+(r)$ . Since  $\lambda_X^+(r) \leq r$  for any  $r \in [0, 1]$ , then  $\lambda \leq \frac{1}{2}$ . Let  $\mu \in (0, \lambda)$ . By the Definitions 1, 2 there exist  $x, y \in \partial\mathfrak{B}_1(o)$  such that  $y^\top x$  and  $\lambda_X(x, y, r) = \mu' \in (\mu, \lambda)$ , and consequently  $\|x + ry - \mu'x\| = 1$ . Since  $y^\top x$  there exists  $p \in J_1(x) = J_1(x - \mu'x)$  such that  $\langle p, y \rangle = 0$ .

Using Lemma 3, we get

$$\begin{aligned} 1 = \|x + ry - \mu'x\| &\leq \|x - \mu'x\| + \langle p, ry \rangle + 2(1 - \mu')\rho_X\left(\frac{r}{1 - \mu'}\right) = \\ &= 1 - \mu' + 2(1 - \mu')\rho_X\left(\frac{r}{1 - \mu'}\right). \end{aligned}$$

To complete the proof, it suffices to note that  $\mu' < \frac{1}{2}$ ,  $\rho_X(0) = 0$  and the modulus of smoothness is a convex function.

□

**Lemma 7.** *Let  $r \in [0, 1]$ . Then*

$$(7) \quad \rho_X\left(\frac{r}{2}\right) \leq \lambda_X^+(r).$$

**Proof.**

Taking into account the definition of the modulus of smoothness, it follows that for any  $\tau \in [0, \frac{1}{2}]$  and  $\varepsilon \in [0, \rho_X(\tau))$  there exist  $x$  and  $y$  such that the following inequality is true

$$(8) \quad \|x + \tau y\| + \|x - \tau y\| - 2 \geq 2(\rho_X(\tau) - \varepsilon).$$

Without loss of generality, we can assume that  $\|x + \tau y\| \geq \|x - \tau y\|$  (hence  $\|x + \tau y\| \geq 1$ ). Denote  $u = \frac{x + \tau y}{\|x + \tau y\|}$ ,  $v = \frac{x - \tau y}{\|x - \tau y\|}$ . By Lemma 4, we obtain

$$(9) \quad \|u - v\| \leq \frac{4\tau}{\|x + \tau y\|};$$

By the triangle inequality, we get

$$\begin{aligned} \|u + v\| &\leq \frac{2\|x\|}{\|x + \tau y\|} + \|x - \tau y\| \left| \frac{1}{\|x + \tau y\|} - \frac{1}{\|x - \tau y\|} \right| \\ &= 2 - \frac{1}{\|x + \tau y\|} (\|x + \tau y\| + \|x - \tau y\| - 2). \end{aligned}$$

Now, by inequality (8), we have that

$$(10) \quad \|u + v\| \leq 2 - \frac{2(\rho_X(\tau) - \varepsilon)}{\|x + \tau y\|}.$$

Let us consider the plane  $ouv$ . By  $\omega$  denote a point lying on the smallest arc  $uv$  of the unit circle such that the supporting line to the unit ball at  $\omega$  is parallel to  $uv$ . Obviously, either  $\lambda_X\left(\omega, \frac{u-v}{\|u-v\|}, \frac{\|u-v\|}{2}\right) \geq 1 - \frac{\|u+v\|}{2}$  or  $\lambda_X\left(\omega, -\frac{u-v}{\|u-v\|}, \frac{\|u-v\|}{2}\right) \geq 1 - \frac{\|u+v\|}{2}$ , i.e.  $\lambda_X^+\left(\frac{\|u-v\|}{2}\right) \geq 1 - \frac{\|u+v\|}{2}$ . Combining this with inequalities (10), we get

$$\frac{2(\rho_X(\tau) - \varepsilon)}{\|x + \tau y\|} \leq 2\lambda_X^+\left(\frac{\|u - v\|}{2}\right).$$

Now, by inequality (9), we obtain

$$\frac{2}{\|x + \tau y\|} (\rho_X(\tau) - \varepsilon) \leq 2\lambda_X^+\left(\frac{2\tau}{\|x + \tau y\|}\right) \leq \frac{2}{\|x + \tau y\|} \lambda_X^+(2\tau).$$

Multiplying both sides by  $\frac{\|x + \tau y\|}{2}$  and passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain (7).

□

**Remark 2.** *By Lemma 7 and the properties of the modulus of smoothness, it follows that  $\lambda_X^+(r) > 0$  for all  $r > 0$ .*

By Lemmas 6, 7 and the properties of the modulus of smoothness we have the following result.

**Theorem 2.** *Let  $X$  be an arbitrary Banach space. Then  $\lambda_X^+(\tau) \asymp \rho_X(\tau)$  as  $\tau \rightarrow 0$  and for any  $r \in [0, \frac{1}{2}]$ :*

$$\rho_X\left(\frac{r}{2}\right) \leq \lambda_X^+(r) \leq \rho_X(2r).$$

## 5. COMPARISON WITH THE BANAS MODULUS

In the paper [1] J. Banaś defined and studied some new modulus of smoothness. Namely, he defined

$$\delta_X^+(\varepsilon) = \sup \left\{ 1 - \frac{\|x + y\|}{2} : x, y \in \mathfrak{B}_1(o), \|x - y\| \leq \varepsilon \right\}, \quad \varepsilon \in [0, 2].$$

The function  $\delta_X^+(\cdot)$  is called the *Banaś modulus*. In the papers [1, 2, 3, 4] several interesting results concerning this modulus were obtained. Particulary, in [1], J. Banaś proved that a space is uniformly smooth iff  $\frac{\delta_X^+(\varepsilon)}{\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . However, from the definition a space is uniformly smooth if and only if  $\frac{\rho_X(\varepsilon)}{\varepsilon} \rightarrow 0$  as  $\varepsilon \rightarrow 0$ . This leads to the question: are the modulus of smoothness and the modulus of Banaś equivalent at zero? It is easy to check that there exist positive constant  $a, b$  such that  $\delta_X^+(t) \leq a\rho_X(bt)$ , but the lower estimate of the modulus of Banaś in terms of the modulus of smoothness is unknown. In the next theorem we prove that the modulus of Banaś and the modulus of supporting smoothness are equivalent at zero, so Theorem 2 answers the above question.

**Theorem 3.** *Let  $X$  be an arbitrary Banach space. Then  $\delta_X^+(\varepsilon) \asymp \delta_X(\varepsilon)$  as  $\varepsilon \rightarrow 0$  and the following inequalities hold:*

$$(11) \quad \delta_X^+(2r) \leq \lambda_X^+(r) \quad \forall r \in [0, 1];$$

$$(12) \quad \lambda_X^+(r) \leq 2\delta_X^+(3r) \quad \forall r \in \left[0, \frac{2}{3}\right].$$

### Proof.

1) First we shall prove inequality (11) for  $r \in [0, 1]$ .

Let  $a, b$  be points of the unit sphere such that  $\|a - b\| \leq 2r$ . By  $X_2$  denote the plane  $aob$ .

There exists a point  $y_2$  of the unit sphere of the plane  $X_2$  such that the supporting line  $l_2$  to the unit ball at this point is parallel to  $ab$ . By definition, put  $y_1 = oy_2 \cap ab$ . There exists a point  $a_2$  in the projection of the point  $a$  on  $l_2$  such that the segments  $y_1y_2, aa_2$  are equal in length and parallel. The point  $b_2$  is defined in the same way, such that  $y_1y_2$  and  $bb_2$  are parallel (see Fig. 6). Without loss of generality we assume that  $\|y_2a_2\| \leq r < 1$ . Since the modulus of supporting smoothness is an increasing function, we have  $\|y_1y_2\| = \|aa_2\| \leq \lambda_X^+(y_2, \|y_2a_2\|) \leq \lambda_X^+(y_2, r)$ . Taking the supremum, we obtain inequality (11).

Taking into account that the modulus of Banaś is a continuous and increasing function, we obtain inequality (11) for  $r = 1$ .

2) Let us prove inequality (12).

By the definition of modulus of supporting smoothness for any  $\varepsilon \in (0, \lambda_X^+(r))$  there exist

- a point  $x \in \partial\mathfrak{B}_1(o)$ ;
- a line  $\ell_1$  supporting to the unit ball at point  $x$ ;



– a point  $y$  on  $\ell_1$  and a point  $z \in \partial\mathfrak{B}_1(o)$  such that

$$\|xy\| = r, \|yz\| > 0, zy \parallel ox \text{ and } \lambda^+\left(x, \frac{xy}{\|xy\|}, r\right) = \|yz\| > \lambda_X^+(r) - \varepsilon > 0.$$

Let  $\ell_2$  be a line parallel to  $\ell_1$  such that  $z \in \ell_2$ . Let  $z, z_1$  be points of the intersections of line  $\ell_2$  and  $\partial\mathfrak{B}_1(o)$ . By  $y_1$  denote the projections of  $z_1$  on  $\ell_1$  such that  $z_1y_1 \parallel ox$  (see Fig. 4).

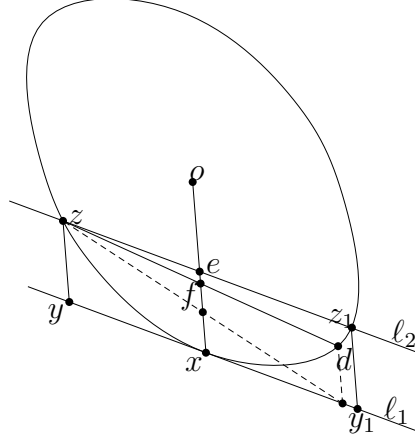


FIGURE 4. Illustration for the second part of Theorem 3.

We shall prove that  $\|zz_1\| \geq 2r$ . In the converse case,  $\|xy_1\| < r$ . Note that if we fix  $x, y \in \partial\mathfrak{B}_1(o)$  such that  $y \perp x$ , then the function  $\lambda^+(x, y, \cdot)$  is strictly increasing on the set of its positive values. Since  $xy_1$  and  $xy$  lie on the same line and by to the definition of  $\lambda^+$ , we obtain

$$\lambda^+\left(x, \frac{xy}{\|xy\|}, r\right) = \|yz\| = \|y_1z_1\| \leq \lambda^+\left(x, \frac{xy_1}{\|xy_1\|}, \|xy_1\|\right) < \lambda^+\left(x, \frac{xy_1}{\|xy_1\|}, r\right) = \lambda^+\left(x, \frac{xy}{\|xy\|}, r\right).$$

Contradiction. Consequently  $\|zz_1\| \geq 2r$ .

By definition, put  $e = ox \cap zz_1$ . By the continuity reasons there exists a point  $d$  on the arc  $z_1x$  of the unit sphere such that for the point  $f = zd \cap ox$  the following equality holds  $\|d - f\| = \|f - z\|$ . Since  $\ell_1$  is a supporting line to the unit sphere, we have  $\|xf\| \geq \frac{\|yz\|}{2}$ . Note that  $\|dz\| \leq 2(\|ze\| + \|ef\|) \leq 3r$ . Combining the last two inequalities, we get

$$\delta_X^+(3r) \geq \|xf\| \geq \frac{\lambda_X^+(r) - \varepsilon}{2}.$$

Passing to the limit as  $\varepsilon \rightarrow 0$ , we obtain inequality (12).

□

From Theorems 2 and 3 we have the following corollary.

**Corollary 1.** *Let  $X$  be an arbitrary Banach space, then  $\delta_X^+(\varepsilon) \asymp \rho_X(\varepsilon)$  as  $\varepsilon \rightarrow 0$  and the following inequalities hold:*

$$\frac{1}{2}\rho_X\left(\frac{r}{6}\right) \leq \delta_X^+(r) \leq \rho_X(r), \quad r \in \left[0, \frac{1}{2}\right].$$

The Day-Nordlander theorem (see [7]) asserts that  $\delta_X(\varepsilon) \leq \delta_H(\varepsilon)$  for  $\varepsilon \in [0, 2]$ , where  $H$  denotes an arbitrary Hilbert space. On the other hand, repeating the arguments from the paper [7] we can show that for any Banach space the following estimate is true  $\delta_H^+(\varepsilon) \leq \delta_X^+(r)$  for

$\varepsilon \in [0, 2]$ . From this and Theorems 2, 3 we obtain a Day–Nordlander type result for the moduli of supporting convexity and supporting smoothness:

**Corollary 2.** *Let  $X$  be an arbitrary Banach space. Then*

$$\lambda_X^-(r) \leq \lambda_H^-(r) = 1 - \sqrt{1 - r^2} = \lambda_H^+(r) \leq \lambda_X^+(r) \quad \forall r \in [0, 1].$$

*If at least one of these inequalities turns into equality, then  $X$  is a Hilbert space.*

## 6. ESTIMATES FOR LIPSCHITZ CONSTANT FOR THE METRIC PROJECTION ONTO A HYPERPLANE

Let us introduce the following characteristic of a space:

$$\xi_X = \sup_{\substack{\|x\|=1, \\ \|y\|=1}} \sup_{p \in J_1(y)} \|x - \langle p, x \rangle y\|.$$

Note that if  $y \in \partial \mathfrak{B}_1(0)$ ,  $p \in J_1(y)$ , then the vector  $(x - \langle p, x \rangle y)$  is a metric projection of  $x$  onto the hyperplane  $H_p = \{x \in X : \langle p, x \rangle = 0\}$ . So,  $\xi_X = \sup_{y \in \mathfrak{B}_1(o)} \sup_{p \in J_1(y)} \xi_X^p$ , where  $\xi_X^p$  is half of diameter of a unit ball's projection onto the hyperplane  $H_p$ . Therefore,  $\xi_X$  is the maximal value of the Lipschitz constant for the metric projection operator onto a hyperplane. Obviously,  $\xi_X \leq 2$  and  $\xi_H = 1$  for a Hilbert space  $H$ .

**Theorem 4.** *For any Banach space  $X$  the following inequality is true:*

$$(13) \quad \frac{1}{1 - \lambda_X^-\left(\frac{1 - \lambda_X^-(1)}{2}\right)} \leq \xi_X \leq \frac{1}{1 - \lambda_X^+\left(\frac{1 - \lambda_X^-(1)}{2}\right)}.$$

**Proof.**

First let us introduce some notation. Let  $x_0$  be an arbitrary point on the unit sphere. Let  $l$  be a supporting line to the unit ball at the point  $x_0$ . Define  $l_2$  as the line such that the following conditions hold:

- a)  $l_2 \parallel ox_0$ ;
- b)  $l_2 \cap l \neq \emptyset$ , by definition, put  $x_2 = l_2 \cap l$ ;
- c)  $l$  is supporting line to the unit ball at some point  $y_2$ ;
- d)  $\|y_2 x_2\| \leq 1$ .

Let  $x_1$  be a point on segment  $x_0 x_2$  such that  $\|x_0 x_1\| = 1$ , let  $l_1$  be a line such that  $x_1 \in l_1$  and  $l_1 \parallel ox_0$ . By definition, put  $y_1$  as the intersection point of line  $l_1$  and the segment  $oy_2$ . Let  $b$  be a point on  $\partial \mathfrak{B}_1(o)$  such that the segment  $ob$  is parallel to  $x_0 x_1$ . By construction, we have that  $x_0 x_1 b o$  is a parallelogram, therefore  $b \in l_1$  and  $y_1 \in x_1 b$ . Let  $a$  be the intersection point of the line  $l_1$  and the unit sphere such that  $a \in x_1 y_1$ .

From the intercept theorem, we have  $\frac{\|x_0 x_2\|}{\|oy_2\|} = \frac{\|x_0 x_1\|}{\|oy_1\|}$ . Therefore

$$(14) \quad \|x_0 x_2\| = \frac{1}{\|oy_1\|} = \frac{1}{1 - \|y_1 y_2\|}.$$

Since  $x_0 x_1 b o$  is a parallelogram, we get  $\|x_1 b\| = \|ox_0\| = 1$ . By construction we have that  $\|x_0 x_1\| = 1$ . Therefore,

$$(15) \quad \|ab\| \leq 1 - \lambda_X^-(1).$$

$$(16) \quad \|y_1 y_2\| \leq \lambda_X^+(\min\{\|a_2 y_2\|, \|y_2 b_2\|\}) \leq \lambda_X^+\left(\frac{\|ab\|}{2}\right) \leq \lambda_X^+\left(\frac{1 - \lambda_X^-(1)}{2}\right).$$

Let  $\varepsilon$  be an arbitrary positive real number. Note that we could choose a point  $x_0$  such that  $\|x_1 a\| \leq \lambda_X^-(1) + \varepsilon$ , i.e.  $\|ab\| \geq 1 - \lambda_X^-(1) - \varepsilon$ . Like in (16), we obtain

$$\|y_1 y_2\| \geq \lambda_X^-(\max\{\|a_2 y_2\|, \|y_2 b_2\|\}) \geq \lambda_X^-\left(\frac{\|ab\|}{2}\right) \geq \lambda_X^-\left(\frac{1 - \lambda_X^-(1) - \varepsilon}{2}\right).$$

9

**Conjecture 1.** *The right-hand side of inequality (13) becomes an equality in case of  $L_p$ ,  $p \in (1; +\infty)$ .*

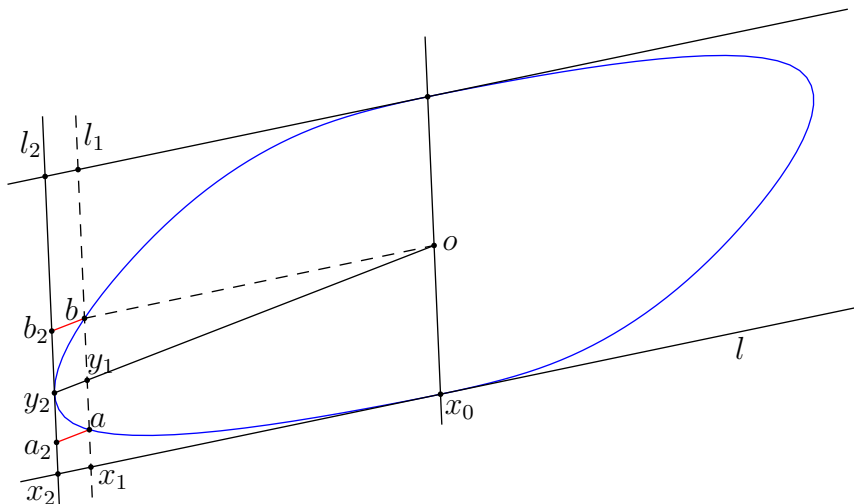


FIGURE 5. Illustration for Theorem 4.

**Lemma 8.** *For any  $r \in [0, 1]$  the following inequalities hold:*

$$(17) \quad 1 - \frac{1}{2}\delta_X^{-1}\left(1 - \frac{r}{2}\right) \leq 1 - \frac{1}{2}\delta_X^{-1}\left(1 - \frac{r}{\xi_X}\right) \leq \lambda_X^+(r).$$

**Proof.**

The left-hand side of inequality (17) is a straightforward consequence of the inequality  $\xi_X \leq 2$ . Let us prove the right-hand side of inequality (17). In case of  $r = 0$  it is trivial. Let  $x_0$  be an arbitrary point on the unit sphere. Define  $H_x$  as a supporting hyperplane to the unit ball at the point  $x_0$ . Let  $x_1$  be a point of the supporting hyperplane  $H_x$  such that  $\|x_0x_1\| = r$ . Denote the ray  $\{ox_0 + \alpha x_0x_1 : \alpha \geq 0\}$  as  $\ell$ . Let  $l_1, l_2$  be the lines parallel to  $ox_0$  such that

- a)  $l_2$  is a supporting line to the unit ball at the point  $y_2$  and  $l_2 \cap \ell = x_2$ ;
- b)  $l_1$  intersects the ray  $\ell$  at  $x_1$  and intersects the unit sphere at points  $a, b$ .

Let  $y_1 = oy_2 \cap ab$  (see Fig. 6). By the definition of  $\lambda_X^+(r)$  and since the unit ball is centrally

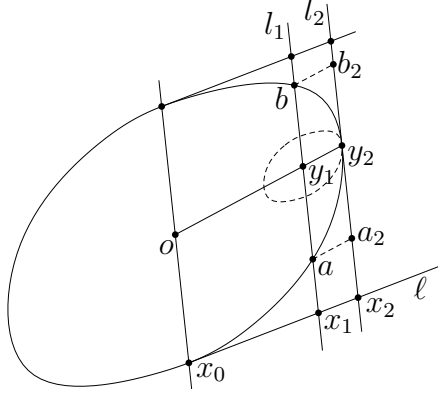


FIGURE 6. Illustration for Lemma 8 and for the first part of Theorem 3.

symmetric, we get  $\|ab\| \geq 2(1 - \lambda_X^+(r))$ . Obviously,  $\|y_1y_2\| \geq \delta_X(\|ab\|)$ . Consequently, we have

$$(18) \quad \delta_X(2(1 - \lambda_X^+(r))) \leq \delta_X(\|ab\|) \leq \|y_1y_2\|.$$

Using the intercept theorem, we obtain

$$(19) \quad \|y_1y_2\| = \frac{\|y_1y_2\|}{\|oy_2\|} = \frac{\|x_1x_2\|}{\|x_0x_2\|} = \frac{\|x_0x_2\| - \|x_0x_1\|}{\|x_0x_2\|} = 1 - \frac{r}{\|x_0x_2\|} \leq 1 - \frac{r}{\xi_X}.$$

By inequalities (18) and (19), we have

$$\delta_X(2(1 - \lambda_X^+(r))) \leq 1 - \frac{r}{\xi_X}$$

□

It is easy to check that in a Hilbert space  $H$  the following equality holds

$$\delta_H^{-1}(\tau) = 2\sqrt{1 - (1 - \tau)^2}.$$

Substituting this in inequality (17) and since  $\xi_H = 1$ , we obtain

$$\delta_H(2r) = 1 - \frac{1}{2}\delta_H^{-1}(1 - r) \leq \lambda_H^+(r).$$

According to (1), we have that if  $X$  is a Hilbert space, then the right hand estimate in inequality (17) is reached.

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